9.2 Perimeter and Area Relationships of a Rectangle

Brandon works during the summers for a fencing company. He has 32 sections of prefabricated fencing, each 1 m in length, to enclose a rectangular area for a customer. The customer wants the enclosure to have the greatest area possible.

Finding the dimensions that will maximize a rectangular area given its perimeter is called *optimizing* the area. The mathematical process used to solve this type of problem is known as **optimization**.

optimization
- the process of finding values that make a given quantity the greatest (or least) possible given certain conditions

Investigate A

How can you model the maximum area of a rectangle with a fixed perimeter?

Tools
- toothpicks or grid paper

Brandon needs to find the dimensions that will maximize the rectangular area of an enclosure with a perimeter of 32 m.

Method 1: Use Manipulatives or Diagrams

1. Use 32 toothpicks, each representing a 1-m section of fencing, to create rectangles of different shapes with whole-number dimensions and a perimeter of 32 m. Or, draw different rectangles on grid paper, letting each grid unit represent 1 m of fencing.

2. Copy and complete the table.

Rectangle	Width (m)	Length (m)	Perimeter (m)	Area (m^2)
1	1	15	32	15
2	2	14	32	
3	3	13		

maximum
- greatest possible

3. **a)** What are the dimensions of the rectangle with **maximum**, or *optimal*, area?
   b) What is the maximum area?
   c) Describe the shape of the rectangle.

4. Suppose the customer decides to use 40 m of fencing instead.
   a) Predict the dimensions of the rectangle with the maximum area.
   b) Use 40 toothpicks or grid paper to test your hypothesis.
   c) Compare your results with your prediction.

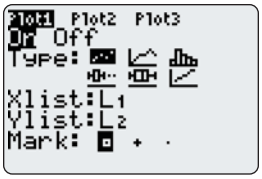
5. **Reflect** How can you predict the dimensions of a rectangle with maximum area if you know the perimeter?

6. a) Use your method to predict the dimensions of a rectangle with maximum area made of 60 toothpicks.
   b) Repeat part a) with 30 toothpicks. Describe any problems in applying your method.

Method 2: Use a Graphing Calculator

Tools
- graphing calculators

1. Use your table of results from step 2 of Method 1 to create a scatter plot of area versus width.
   a) Enter the width values in list **L1** and the area values in list **L2**.
   b) Set up the scatter plot using the settings shown.
   c) Press ZOOM and select **9:Zoomstat** to create an appropriate window.



2. Draw a sketch of the scatter plot in your notebook. Circle the region where the area of the garden is the greatest.

3. You can create an algebraic model of this relationship. Look at the columns entitled Width and Length in your table of results. Notice that the length and width values always have a sum of 16.
   a) Consider a rectangle of width x. Explain why the length of the rectangle can be represented by the expression $(16 - x)$.
   b) The area of a rectangle is the product of its length and width. Use the expressions for length and width to create an expression for the area.

4. Press Y=, and then enter $x(16 - x)$. Press GRAPH. How does this graph compare to the scatter plot?

5. Turn off Plot1. Use the TRACE feature on the graphing calculator to trace points on the graph of $y = x(16 - x)$.
   a) Find the point on the graph that represents the rectangle with maximum area. The x-coordinate of this point represents the width of the rectangle and the y-coordinate represents the area of the rectangle.
   b) Record the width and the area of the rectangle. Verify that this is the same rectangle you found in Method 1. Describe the location of this point on the curve.

6. **a)** How does the algebraic model change if the perimeter of the rectangle is 40 m? Graph the relationship. Locate the point on the graph that represents the rectangle with maximum area.

   b) What are the dimensions of the rectangle with maximum area? Describe the shape of the rectangle. Does this agree with your findings in step 4 of Method 1?

7. Suppose the perimeter of a rectangle is 45 m.

   a) Predict the dimensions of the rectangle with maximum area.

   b) Check your prediction by creating an algebraic model and graphing it. Was your prediction correct?

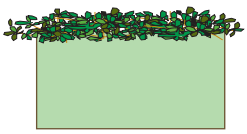
8. **Reflect** How can you predict the dimensions of a rectangle with maximum area if you know the perimeter?

Investigate B

Tools

- toothpicks or grid paper

How can you model the maximum area of a rectangle with a fixed sum of the lengths of three sides?



Brandon's customer decides to use an existing hedge as one of the boundaries for the enclosure. This means that he will only use the prefabricated fencing on three sides of the rectangular enclosure. The client still wants the enclosure to have the greatest area possible.

1. Brandon has 32 m of prefabricated fencing.

   a) Do you think Brandon will be able to enclose more, less, or the same amount of area now that the hedge is being used on one side?

   b) What shape do you think will have the maximum area?

   c) Make a hypothesis about what dimensions will have the maximum area.

2. Use toothpicks or sketch rectangles on grid paper to determine the dimensions of the rectangle that has the maximum area. Record your results in a table.

Rectangle	Width (m)	Length (m)	Sum of Lengths of Three Sides (m)	Area (m^2)
1	1	30	32	30
2	2	28	32	
3	3			

3. **a)** What are the dimensions of the rectangle with maximum area?

   b) Compare this result with your hypothesis.

c) Compare this result with the maximum area you found in Investigate A. Will the hedge allow Brandon to enclose more, less, or the same amount of area as before?

4. Examine the length and width of the enclosure of maximum area that you found. Describe any relationship that you notice.

5. Suppose Brandon has 40 m of prefabricated fencing to work with.

a) Predict the dimensions of the rectangle with maximum area.

b) Test your prediction. Were you correct? Do you need to change your hypothesis?

6. **Reflect** How can you predict the dimensions of a rectangle with maximum area if you know the sum of the lengths of three sides?

Key Concepts

- Optimizing the area of a rectangle means finding the dimensions of the rectangle with maximum area for a given perimeter.
- For a rectangle with a given perimeter, there are dimensions that result in the maximum area.
- The dimensions of a rectangle with optimal area depend on the number of sides to be fenced. If fencing is not required on all sides, a greater area can be enclosed.

Communicate Your Understanding

C1 A farmer wants to fence a rectangular field. Suggest two things that will allow the farmer to maximize the enclosed area.

C2 **a)** When does a square maximize the enclosed area?

b) When does a square not maximize the enclosed area?

C3 At a lake, a rectangular swimming area is to be roped in on three sides to create the greatest area possible. How will the length and width of this optimal area be related?

Practise

1. What dimensions will provide the maximum area for a rectangle with each perimeter?

a) 20 m **b)** 36 m

c) 50 m **d)** 83 m

Connect and Apply

2. To brighten a room, a rectangular window will be built into a wall. To keep the cost as low as possible, the perimeter of the window must be 6.0 m.

a) Sketch three different windows that have a perimeter of 6.0 m. Include dimensions.

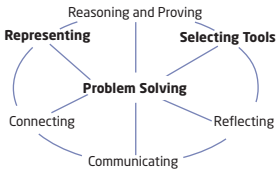
b) What window dimensions will allow the maximum amount of light to enter the room?

3. A rectangular enclosure is to be created using 82 m of rope.

a) What are the dimensions of the rectangle of maximum area?

b) Suppose 41 barriers, each 2 m long, are used instead. Can the same area be enclosed? Explain.

c) How much more area can be enclosed if the rope is used instead of the barriers?



4. A farmer is adding a rectangular corral to the side of a barn. The barn will form one side of the rectangle. The farmer has 16 m of fencing to use. Conduct an investigation to determine the dimensions of the corral with maximum area. Use any tools: toothpicks, geoboards, grid paper, tables, or technology such as spreadsheets, *The Geometer's Sketchpad*®, or a graphing calculator.

5. A fence is to be built with prefabricated sections that are 2.8 m in length. What is the maximum rectangular area that you can enclose with

a) 20 pieces?

b) 40 pieces?

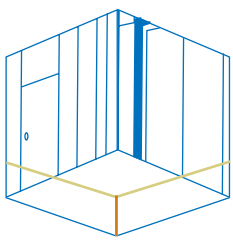
6. A fence is being built using the materials in question 5, but now there is an existing wall that will be used as one of the boundaries. Draw a diagram and label the dimensions of the maximum rectangular area that you can enclose with

a) 20 pieces

b) 40 pieces

For the fence materials in each of parts a) and b), how much additional area does using an existing border provide?

7. **Chapter Problem** Talia's uncle owns a warehouse and he has given Talia an area in which to store the computer supplies for his company. Her uncle gave Talia 40 m of rope and told her to section off a rectangular area in a corner of the warehouse. Conduct an investigation to determine the greatest area that Talia can rope off.

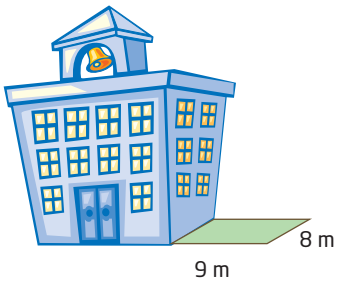


8. Brandon prepares a proposal for his client.
In the proposal, he reports how the 32 m of fencing that is available can be used to fence an enclosure on

- four sides
- three sides, using a hedge at the back of the property as the fourth side
- two sides, using the hedge and an existing neighbour's fence on an adjacent side

Draw diagrams for each of the three scenarios in Brandon's proposal and calculate the maximum area that can be enclosed in each case.

9. A contractor is adding a rectangular kindergarten playground to the side of a school. The school will form one side of the rectangle. The area of the playground is to be 72 m^2. One possible rectangle is shown.



a) Investigate other possible rectangles with an area of 72 m^2. Copy and complete the table or use a spreadsheet like the one shown.

Rectangle	Width (m)	Length (m)	Area (m^2)	Length of Fence Used (m)
1			72	
2			72	
3			72	

	A	B	C	D	E
1	Rectangle	Width (m)	Length (m)	Area (m^2)	Length of Fence Used (m)
2	1	1	=72/B2	72	=C2+2*B2
3	2	=B2+1	=72/B3	72	=C3+2*B3
4					

b) What dimensions use the minimum length of fence to enclose the playground?

c) What is the minimum length of fence that can be used to enclose the playground?

10. Pose a problem involving the relationship between the perimeter and the area of a garden. Solve the problem and then have a classmate solve it.

11. Describe a situation in which it is important to know

a) the minimum perimeter of a rectangle for a given area

b) the maximum area of a rectangle for a given perimeter

Achievement Check

12. A rectangular area is to be enclosed with 12 m of fencing.

a) What is the maximum area that can be enclosed if fencing is used on all four sides? What are the dimensions of this optimal shape?

b) Suppose an existing hedge is used to enclose one side. Use diagrams or toothpicks to determine the maximum area that can be enclosed. Record your results in a table. What are the dimensions of this shape?

c) Suppose two perpendicular hedges enclose the area on two sides. What are the dimensions of the maximum area that can be enclosed?

Extend

Did You Know?

According to the Building Code Act of Ontario, a building permit is required for any new building with area greater than 10 m^2.

13. Conduct an investigation to determine the dimensions of the rectangular floor of a toolshed with area 35 m^2 and minimum perimeter.

14. A rectangular yard with an area of 50 m^2 is to be fenced on three sides. Minimizing the perimeter will minimize the cost of the fence. Conduct an investigation to determine the shape of the yard with minimum perimeter.

15. If a triangle is drawn inside a circle so that the three vertices touch but do not intersect the circle boundary, then the triangle is *inscribed* in the circle. Conduct an investigation to find the dimensions of the triangle of maximum area that can be inscribed in a circle with diameter 20 cm.

16. **Math Contest** Find the dimensions of the rectangle of maximum area that can be inscribed in a circle of radius 10 cm.

17. **Math Contest** Katrina and Ranjeet have a piece of string 24 cm long and want to determine the maximum area that can be enclosed by the string. Katrina said, "The shape that will give the maximum area is a square. So, the maximum area for a square with sides 6 cm each is 36 cm^2." Ranjeet replied, "I can make a figure with a greater area." Is Ranjeet correct? If so, find the maximum area that can be enclosed with the string.

18. **Math Contest** A farmer has 500 m of fencing. He wants to construct three adjoining rectangular fields that have the greatest possible area. Determine the dimensions of the three fields.